19. Isaac Newton Helps Samuel Pepys

Pepys wrote Newton to ask which of three events is more likely: that a person get (a) at least 1 six when 6 dice are rolled, (b) at least 2 sixes when 12 dice are rolled, or (c) at least 3 sixes when 18 dice are rolled. What is the answer?

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A, *B*, and *C* are to fight a three-cornered pistol duel. All know that *A*'s chance of hitting his target is 0.3, *C*'s is 0.5, and *B* never misses. They are to fire at their choice of target in succession in the order *A*, *B*, *C*, cyclically (but a hit man loses further turns and is no longer shot at) until only one man is left unhit. What should *A*'s strategy be?

21. Should You Sample with or without Replacement?

Two urns contain red and black balls, all alike except for color. Urn *A* has 2 reds and 1 black, and Urn *B* has 101 reds and 100 blacks. An urn is chosen at random, and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first ball is drawn and its color reported, you can decide whether or not the ball shall be replaced before the second drawing. How do you order the second drawing, and how do you decide on the urn?

22. The Ballot Box

In an election, two candidates, Albert and Benjamin, have in a ballot box a and b votes respectively, $a > b$, for example, 3 and 2. If ballots are randomly drawn and tallied, what is the chance that at least once after the first tally the candidates have the same number of tallies?

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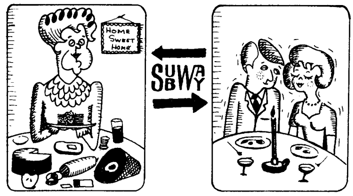
Players *A* and *B* match pennies *N* times. They keep a tally of their gains and losses. After the first toss, what is the chance that at no time during the game will they be even?



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Marvin gets off work at random times between 3 and 5 P.M. His mother lives uptown, his girl friend downtown. He takes the first subway that comes in either direction and eats dinner with the one he is first delivered to. His mother complains that he never comes to see her, but he says she has a 50-50 chance. He has had dinner with her twice in the last 20 working days. Explain.

25. Lengths of Random Chords

If a chord is selected at random on a fixed circle, what is the probability that its length exceeds the radius of the circle?

26. The Hurried Duelers

Duels in the town of Discretion are rarely fatal. There, each contestant comes at a random moment between 5 A.M. and 6 A.M. on the appointed day and leaves exactly 5 minutes later, honor served, unless his opponent arrives within the time interval and then they fight. What fraction of duels lead to violence?

27. Catching the Cautious Counterfeiter

(a) The king's minter boxes his coins 100 to a box. In each box he puts 1 false coin. The king suspects the minter and from each of 100 boxes draws a random coin and has it tested. What is the chance the minter's peculations go undetected?

(b) What if both 100's are replaced by n?

Consequently, the probability of an even split is

$$P(\text{even split}) = \frac{100!}{50!50!}\left(\frac{1}{2}\right)^{100}.$$

Evaluating this with logarithms, I get 0.07959 or about 0.08.

Stirling's Approximation

Sometimes, to work theoretically with large factorials, we use Stirling's approximation

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n},$$

where e is the base of the natural logarithms. The percentage error in the approximation is about $100/12n$. Let us use Stirling's approximation on the probability of an even split

$$P(\text{even split}) \approx \frac{\sqrt{2\pi}\,100^{100+\frac{1}{2}}e^{-100}}{(\sqrt{2\pi}\,50^{50+\frac{1}{2}}e^{-50})^2 2^{100}} = \frac{100^{\,100+\frac{1}{2}}}{\sqrt{2\pi}\,50^{100}50(2^{100})}$$

$$= \frac{\sqrt{100}}{\sqrt{2\pi}\,50} = \frac{1}{\sqrt{50\pi}} = \frac{1}{5\sqrt{2\pi}}.$$

Since $1/\sqrt{2\pi}$ is about 0.4, the approximation gives about 0.08 as we got before. More precisely the approximation gives to four decimals 0.0798 instead of 0.0796.

Stirling's approximation is discussed in advanced calculus books. For one nice treatment see R. Courant, *Differential and integral calculus*, Vol. I, Translated by E. J. McShane, Interscience Publishers, Inc., New York, 1937, pp. 361–364.

19. Isaac Newton Helps Samuel Pepys

Pepys wrote Newton to ask which of three events is more likely: that a person get (a) at least 1 six when 6 dice are rolled, (b) at least 2 sixes when 12 dice are rolled, or (c) at least 3 sixes when 18 dice are rolled. What is the answer?

Solution for Isaac Newton Helps Samuel Pepys

Yes, Samuel Pepys wrote Isaac Newton a long, complicated letter about a wager he planned to make. To decide which option was the favorable one, Pepys needed the answer to the above question. You may wish to read the correspondence in *American Statistician*, Vol. 14, No. 4, Oct., 1960, pp. 27–30, "Samuel Pepys, Isaac Newton, and Probability," discussion by Emil D. Schell in "Questions and Answers," edited by Ernest Rubin; and further comment in the issue of Feb., 1961, Vol. 15, No. 1, p. 29. As far as I know this is Newton's only venture into probability.

Since 1 is the average or mean number of sixes when 6 dice are thrown, 2 the average number for 12 dice, and 3 the average number for 18, one might think that the probabilities of the three events must be equal. And many would think it equal to $\frac{1}{2}$. That thought would be another instance of confusion between averages and probabilities. When the number of dice thrown is very large, then the probability that the number of sixes equals or exceeds the expected number is slightly larger than $\frac{1}{2}$. Thus for large numbers of dice, the supposition is nearly true, but not for small numbers. For large numbers of dice, the distribution of the number of sixes is approximately symmetrical about the mean, and the term at the mean is small, but for small numbers of dice, the distribution is asymmetrical and the probability of rolling exactly the mean number is substantial.

Let us begin by computing the probability of getting exactly 1 six when 6 dice are rolled. The chance of getting 1 six and 5 other outcomes in a particular order is $(\frac{1}{6})(\frac{5}{6})^5$. We need to multiply by the number of orders for 1 six and 5 non-sixes. In An Even Split at Coin Tossing, Problem 18, we learned to count the number of orders and we get $\binom{6}{1}$. Therefore the probability of exactly 1 six is

$$\binom{6}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5.$$

Similarly, the probability of exactly x sixes when 6 dice are thrown is

$$\binom{6}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{6-x}, \qquad x = 0, 1, 2, 3, 4, 5, 6.$$

The probability of x sixes for n dice is

$$\binom{n}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{n-x}, \qquad x = 0, 1, \ldots, n.$$

This formula gives the terms of what is called a binomial distribution.

The probability of 1 or more sixes with 6 dice is the complement of the probability of 0 sixes:

$$1 - \binom{6}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 \approx 0.665.$$

When $6n$ dice are rolled, the probability of n or more sixes is

$$\sum_{x=n}^{6n}\binom{6n}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{6n-x} = 1 - \sum_{x=0}^{n-1}\binom{6n}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{6n-x}.$$

Unfortunately, Newton had to work the probabilities out by hand, but we can use the *Tables of the cumulative binomial distribution*, Harvard University Press, 1955. Fortunately, this table gives the cumulative binomial for various values of p (the probability of success on a single trial), and one of the tabled values is $p = \frac{1}{6}$. Our short table shows the probabilities, rounded to three decimals, of obtaining the mean number or more sixes when $6n$ dice are tossed.

$6n$	n	$P(n$ or more sixes)
6	1	0.665
12	2	0.619
18	3	0.597
24	4	0.584
30	5	0.576
96	16	0.542
600	100	0.517
900	150	0.514

Clearly Pepys will do better with the 6-dice wager than with 12 or 18. When he found that out, he decided to welch on his original bet.

The binomial distribution is treated extensively in PWSA, Chapter 7, see especially pp. 241–257.

20. The Three-Cornered Duel

A, *B*, and *C* are to fight a three-cornered pistol duel. All know that *A*'s chance of hitting his target is 0.3, *C*'s is 0.5, and *B* never misses. They are to fire at their choice of target in succession in the order *A*, *B*, *C*, cyclically (but a hit man loses further turns and is no longer shot at) until only one man is left unhit. What should *A*'s strategy be?

Solution for The Three-Cornered Duel

A naturally is not feeling cheery about this enterprise. Having the first shot he sees that, if he hits *C*, *B* will then surely hit him, and so he is not going to shoot at *C*. If he shoots at *B* and misses him, then *B* clearly shoots the more dangerous *C* first, and *A* gets one shot at *B* with probability 0.3 of succeeding. If he misses this time, the less said the better. On the other hand, suppose *A* hits *B*. Then *C* and *A* shoot alternately until one hits. *A*'s chance of winning is

$$(.5)(.3) + (.5)^2(.7)(.3) + (.5)^3(.7)^2(.3) + \cdots.$$

Each term corresponds to a sequence of misses by both *C* and *A* ending with a final hit by A. Summing the geometric series, we get

$$(.5)(.3)\{1 + (.5)(.7) + [(.5)(.7)]^2 + \cdots\}$$
$$= \frac{(.5)(.3)}{1 - (.5)(.7)} = \frac{.15}{.65} = \frac{3}{13} < \frac{3}{10}.$$

Thus hitting B and finishing off with C has less probability of winning for A than just missing the first shot. So A fires his first shot into the ground and then tries to hit B with his next shot. C is out of luck.

In discussing this with Thomas Lehrer, I raised the question whether that was an honorable solution under the code duello. Lehrer replied that the honor involved in three-cornered duels has never been established, and so we are on safe ground to allow A a deliberate miss.

21. Should You Sample with or without Replacement?

Two urns contain red and black balls, all alike except for color. Urn A has 2 reds and 1 black, and Urn B has 101 reds and 100 blacks. An urn is chosen at random, and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first ball is drawn and its color reported, you can decide whether or not the ball shall be replaced before the second drawing. How do you order the second drawing, and how do you decide on the urn?

Solution for Should You Sample with or without Replacement?

If the first ball drawn is a red, then no matter which urn is being drawn from, it now has half red and half black balls, and the second ball provides no discrimination. Therefore if red is drawn first, replace it before drawing again. If black is drawn, do not replace it. When this strategy is followed, the probabilities associated with the outcomes are

	Urn A	Urn B	decide
2 reds	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3}$	$\frac{1}{2}\cdot\frac{101}{201}\cdot\frac{101}{201}\approx\frac{1}{8}$	Urn A
red, then black	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{3}$	$\frac{1}{2}\cdot\frac{101}{201}\cdot\frac{100}{201}\approx\frac{1}{8}$	Urn B
black, then red	$\frac{1}{2}\cdot\frac{1}{3}\cdot 1$	$\frac{1}{2}\cdot\frac{100}{201}\cdot\frac{101}{200}\approx\frac{1}{8}$	Urn A
2 black	$\frac{1}{2}\cdot\frac{1}{3}\cdot 0$	$\frac{1}{2}\cdot\frac{100}{201}\cdot\frac{99}{200}\approx\frac{1}{8}$	Urn B

The total probability of deciding correctly is approximately (replacing $\frac{100}{201}$ by $\frac{1}{2}$, etc.)

$$\tfrac{1}{2}[\tfrac{4}{9} + \tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{4}] = \tfrac{23}{36} \approx 0.64.$$

Drawing both balls *without* replacement gives about 5/8, drawing both *with* replacement gives about 21.5/36.

22. The Ballot Box

In an election, two candidates, Albert and Benjamin, have in a ballot box a and b votes respectively, $a > b$, for example, 3 and 2. If ballots are randomly drawn and tallied, what is the chance that at least once after the first tally the candidates have the same number of tallies?

Solution for The Ballot Box

For $a = 3$ and $b = 2$, the equally likely sequences of drawings are

$A\ A\ A\ B\ B$	$*A\ A\ B\ B\ A$	$*A\ B\ B\ A\ A$
$A\ A\ B\ A\ B$	$*A\ B\ A\ B\ A$	$*B\ A\ B\ A\ A$
$*A\ B\ A\ A\ B$	$*B\ A\ A\ B\ A$	$*B\ B\ A\ A\ A$
$*B\ A\ A\ A\ B$		

where the starred sequences lead to ties, and thus the probability of a tie in this example is $\frac{8}{10}$.

More generally, we want the proportion of the possible tallying sequences that produce at least one tie. Consider those sequences in which the *first* tie appears when exactly $2n$ ballots have been counted $n \leq b$. For every sequence in which A (for Albert) is always ahead until the tie, there is a corresponding sequence in which B (for Benjamin) is always ahead until the tie. For example, if $n = 4$, corresponding to the sequence

$$A\ A\ B\ A\ B\ A\ B\ B$$

in which A leads until the tie, there is the complementary sequence

$$B\ B\ A\ B\ A\ B\ A\ A$$

in which B always leads. This second sequence is obtained from the first by replacing each A by a B and each B by an A.

Given a tie sometime, there is a first one. The number of sequences with A ahead until the first tie is the same as the number with B ahead until the first tie. The trick is to compute the probability of getting a first tie with B ahead until then.

Since A has more votes than B, A must ultimately be ahead. If the first ballot is a B, then there must be a tie sooner or later; and the only way to get a first tie with B leading at first is for B to receive the first tally. The probability that the first ballot is a B is just

$$\frac{b}{a + b}.$$

But there are just as many tie sequences resulting from the first ballot's being an A. Thus the probability of a tie is exactly

$$P(\text{tie}) = \frac{2b}{a + b} = \frac{2}{r + 1},$$

where $r = a/b$. We note that when a is much larger than b, that is, when r gets large, the probability of a tie tends to zero (a result that is intuitively reasonable). And the formula holds when $b = a$, because we must have a tie and the formula gives unity as the probability.

23. Ties in Matching Pennies

Players A and B match pennies N times. They keep a tally of their gains and losses. After the first toss, what is the chance that at no time during the game will they be even?

Solution for Ties in Matching Pennies

Below we extend the method described in the Solution for The Ballot Box, Problem 22, to show that the probability of not getting a tie is (for N odd and N even)

$$P(\text{no tie}) = \binom{N-1}{n} \Big/ 2^{N-1}, \qquad N = 2n + 1,$$

$$P(\text{no tie}) = \binom{N}{n} \Big/ 2^{N}, \qquad N = 2n.$$

The formulas show that the probability is the same for an even N and for the following odd number $N + 1$. For example, when $N = 4$, the second formula applies. The 16 possible outcomes are

*$A\,A\,A\,A$	$B\,A\,A\,A$	$A\,B\,B\,A$	$B\,A\,B\,B$
*$A\,A\,A\,B$	$A\,A\,B\,B$	$B\,A\,B\,A$	*$B\,B\,A\,B$
*$A\,A\,B\,A$	$A\,B\,A\,B$	$B\,B\,A\,A$	*$B\,B\,B\,A$
$A\,B\,A\,A$	$B\,A\,A\,B$	$A\,B\,B\,B$	*$B\,B\,B\,B$

where the star indicates that no tie occurs. Since the number of combinations of 4 things taken 2 at a time is 6, the formula checks.

For $N = 2n$, the probability of x wins for A is $\binom{N}{x} \Big/ 2^N$. If $x \leq n$, the probability of a tie is $2x/N$, based on the ballot box result, and for $x \geq n$ it is $2(N - x)/N$. To get the unconditional probability of a tie, we weight the probability of the outcome x by the probability of a tie with x wins and sum to get

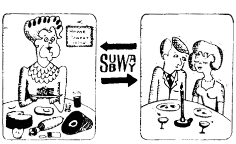
$$(1) \quad 2(2^{-N})\left[\frac{0}{N}\binom{N}{0} + \frac{1}{N}\binom{N}{1} + \cdots + \frac{n-1}{N}\binom{N}{n-1} + \frac{n}{N}\binom{N}{n} + \frac{n-1}{N}\binom{N}{n+1} + \cdots + \frac{1}{N}\binom{N}{N-1} + \frac{0}{N}\binom{N}{N}\right].$$

When the binomial coefficients are converted to factorials and their coefficients canceled, we find that, except for a missing term which is $(N-1)!/n!(n-1)! = \binom{N-1}{n}$, the sum in brackets would be $\sum\binom{N-1}{x}$ over the possible values of x. Consequently, we can rewrite expression (1) as

$$(2) \qquad 2^{-N+1}\left[2^{N-1} - \binom{N-1}{n}\right] = 1 - \binom{N-1}{n} \Big/ 2^{N-1}.$$

The complement of expression (2) gives at last the probability of no tie $\binom{N-1}{n} \Big/ 2^{N-1}$, which a little algebra shows can be written $\binom{N}{n} \Big/ 2^N$ as suggested earlier.

24. The Unfair Subway



Marvin gets off work at random times between 3 and 5 P.M. His mother lives uptown, his girl friend downtown. He takes the first subway that comes in either direction and eats dinner with the one he is first delivered to. His mother complains that he never comes to see her, but he says she has a 50-50 chance. He has had dinner with her twice in the last 20 working days. Explain.

Solution for The Unfair Subway

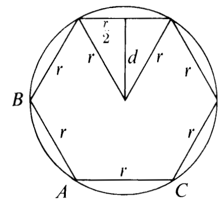
Downtown trains run past Marvin's stop at, say, 3:00, 3:10, 3:20, . . . , etc., and uptown trains at 3:01, 3:11, 3:21, ... To go uptown Marvin must arrive in the 1-minute interval between a downtown and an uptown train.

25. Lengths of Random Chords

If a chord is selected at random on a fixed circle what is the probability that its length exceeds the radius of the circle?

Some Plausible Solutions for Lengths of Random Chords

Until the expression "at random" is made more specific, the question does not have a definite answer. The three following plausible assumptions, together with their three different probabilities, illustrate the uncertainty in the notion of "at random" often encountered in geometrical probability problems.



We cannot guarantee that any of these results would agree with those obtained from some physical process which the reader might use to pick random chords, indeed, the reader may enjoy studying empirically whether any do agree.

Let the radius of the circle be r.

(a) Assume that the distance of the chord from the center of the circle is evenly (uniformly) distributed from 0 to r. Since a regular hexagon of side r can be inscribed in a circle, to get the probability, merely find the distance d from the center and divide by the radius. Note that d is the altitude of an equilateral triangle of side r. Therefore from plane geometry we get $d = \sqrt{r^2 - r^2/4} = r\sqrt{3}/2$. Consequently, the desired probability is

$$r\sqrt{3}/2r = \sqrt{3}/2 \approx 0.866.$$

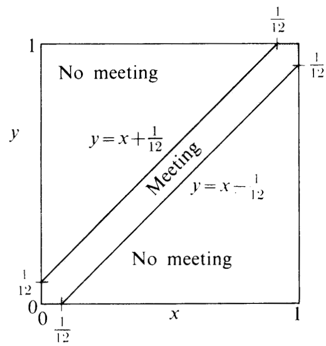
(b) Assume that the midpoint of the chord is evenly distributed over the interior of the circle. Consulting the figure again, we see that the chord is longer than the radius when the midpoint of the chord is within d of the center. Thus all points in the circle of radius d, concentric with the original circle, can serve as midpoints of the chord. Their fraction, relative to the area of the original circle, is $\pi d^2/\pi r^2 = d^2/r^2 = \frac{3}{4} = 0.75$. This probability is the square of the result we got from assumption (a) above.

(c) Assume that the chord is determined by two points chosen so that their positions are independently evenly distributed over the circumference of the original circle. Suppose the first point falls at A in the figure. Then for the chord to be shorter than the radius, the second point must fall on the arc BAC, whose length is $\frac{1}{3}$ the circumference. Consequently, the probability that the chord is longer than the radius is $1 - \frac{1}{3} = \frac{2}{3}$.

26. The Hurried Duelers

Duels in the town of Discretion are rarely fatal. There, each contestant comes at a random moment between 5 A.M. and 6 A.M. on the appointed day and leaves exactly 5 minutes later, honor served, unless his opponent arrives within the time interval and then they fight. What fraction of duels lead to violence?

Solution for The Hurried Duelers



Let x and y be the times of arrivals measured in parts of an hour from 5 A.M. The shaded region of the figure shows the arrival times for which the duelists meet.

The probability that they do not meet is $(\frac{11}{12})^2$, and so the fraction of duels in which they meet is $\frac{23}{144} \approx \frac{1}{6}$.

27. Catching the Cautious Counterfeiter

(a) The king's minter boxes his coins 100 to a box. In each box he puts 1 false coin. The king suspects the minter and from each of 100 boxes draws a random coin and has it tested. What is the chance the minter's peculations go undetected?

(b) What if both 100's are replaced by n?

Solution for Catching the Cautious Counterfeiter

(a) $P(0 \text{ false coins}) = (1 - \frac{1}{100})^{100} \approx 0.366$.

(b) Let there be n boxes and n coins per box.

For any box the chance that the coin drawn is good is $1 - 1/n$, and since there are n boxes,

$$P\ (0 \text{ false coins}) = \left(1 - \frac{1}{n}\right)^n.$$

Let us look at this probability for a few values of n.

n	$P(0 \text{ false coins})$
1	0
2	0.250
3	0.296
4	0.316
5	0.328
10	0.349
20	0.358
100	0.366
1000	0.3677
∞	$0.367879\ldots = 1/e$

Two things stand out. First, the tabled numbers increase; and second, they may be approaching some number. The number they are approaching is well known, and it is e^{-1} or $1/e$, where e is the base of the natural logarithms, 2.71828 . . . .

If we expand $\left(1 - \frac{1}{n}\right)^n$ in powers of $1/n$, we get

$$1^n - \binom{n}{1} 1^{n-1} \left(\frac{1}{n}\right) + \binom{n}{2} 1^{n-2} \left(\frac{1}{n}\right)^2 - \binom{n}{3} 1^{n-3} \left(\frac{1}{n}\right)^3 + \cdots$$

or

$$(1) \qquad 1 - \frac{n}{n} + \frac{n(n-1)}{2!n^2} - \frac{n(n-1)(n-2)}{3!n^3} + \cdots .$$

If we take one of these terms, say the fourth, and study its behavior as n becomes very large, we find that it approaches $-1/3!$ because

$$(2) \qquad \frac{n(n-1)(n-2)}{n^3} = 1\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) = 1 - \frac{3}{n} + \frac{2}{n^2} .$$

As n grows large, all terms on the right-hand side of eq. (2) except the 1 tend to zero. Similarly, for the rth term of expansion (1) the factors depending on n tend to 1, and the term itself tends except for sign to $1/(r-1)!$. Therefore, as n grows, the series for $\left(1 - \frac{1}{n}\right)^n$ tends to

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots .$$

This series is one way of writing e^{-1}.

Had we investigated the case of 2 false coins in every box, we would have found that $\left(1 - \frac{2}{n}\right)^n$ tends to e^{-2} as n grows large, and in general that $\left(1 - \frac{m}{n}\right)^n$ tends to e^{-m}. Also $\left(1 + \frac{m}{n}\right)^n$ tends to e^m whether m is an integer or not. These facts are important for us. They can be studied at more leisure and more rigorously in calculus books, for example, Thomas, G. B., Jr., *Elements of calculus and analytic geometry*, Addison-Wesley, Reading, Mass., 1959, pp. 384–399.

28. Catching the Greedy Counterfeiter

The king's minter boxes his coins n to a box. Each box contains m false coins. The king suspects the minter and randomly draws 1 coin from each of n boxes and has these tested. What is the chance that the sample of n coins contains exactly r false ones?